The space of spheres, a geometric tool to unify duality results on Voronoi diagrams

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The Space of Spheres, a Geometric Tool to Unify Duality Results on Voronoi Diagrams *
L'Espace des Sphères, un outil géométrique unifiant les résultats de dualité sur les diagrammes de Voronoï

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Key words : duality, generalized Voronoi diagrams, geometric transforms, upper envelopes.

Mots clés : dualité, diagrammes de Voronoï généralisés, transformations géométriques, enveloppes supérieures.

Abstract

We present a framework that allows to easily derive transformations for various kinds of Voronoi and Power diagrams including k-th order and weighted versions. All proofs are essentially geometric and require no analytic calculations. Some new algorithms can be presented in special cases.

Résumé

Nous présentons une structure permettant de retrouver facilement de nombreuse transformations sur les diagrammes de Voronoi et de puissance, y compris pour les diagrammes d'ordre supérieur et à poids. Les preuves sont essentiellement de nature géométrique et permettent d'éviter tout calcul analytique. Dans certains cas on peut déduire de nouveaux algorithmes.

1 Introduction

The Voronoi diagram is one of the fundamental problems that computational geometry deals with [PS85,Ede87]. The Voronoi diagram is used to answer to proximity queries : *what is the nearest object from the query point ?* From the original problem where the objects are points in the plane equipped with the Euclidean distance, the problem has been generalized in various directions : both objects and distance can be modified to define other kinds of diagrams ; this problem can also be generalized in higher dimensions. In fact, the Voronoi diagram is not a unique problem, the variety of the possible generalizations has induced a multiplicity of algorithms dealing with the different cases. F. Aurenhammer wrote a very complete survey concerning generalized Voronoi diagrams [Aur91].

A very powerful tool in geometry (as in many domains) is the use of transformations. If Problem A can be transformed into Problem B, all the knowledge of Problem B can be immediatly used for Problem A. This technique has been extensively used for Voronoi diagrams. One of the first result is due to K. Brown [Bro79] who transformed Voronoi diagrams into convex hulls. This result has been followed by many different transformations to handle the diversity of the Voronoi diagrams problems.

Many geometric transforms map the Voronoi diagram in dimension d in an upper envelope of surfaces in dimension $d+1$. We do not claim to give new results on this topic, but we introduce a tool allowing to have a global view and, above all, a better understanding of a very large number of duality results.

For example, F. Aurenhammer [Aur87] introduces the same duality as ours, between spheres and points, for power diagrams, but he gives no geometric interpretation to this, and his justifications are only by analytic computations. We can give new proofs, avoiding all analytic calculations, with only geometric reasoning.

We use here a geometric interpretation of spheres as points in the *space of spheres*, which is a very powerful tool, and will probably find many applications in the future, since it allows to deal with some very general problems in a simple way (see for example Section 4.3 for the case of weighted order k power diagrams, or Section 4.6 for Voronoi diagrams of general manifolds). We can also deduce algorithmic application in some cases.

The paper is organized as follows. Section 2 provides some classical mathematical notions concerning spheres, pencil of spheres, polarity... Mathematicians will probably find this section elementary, and other interested readers can find more details about the mathematic stuff in many classical books. Section 3 deals with the space of spheres and its basic properties. Then Section 4 explains how it can be used for different kinds Voronoi diagrams, and how several generalizations like higher order diagrams, weighted diagrams, diagrams of segments...can be combined together easily. Finally, Section 5 ends with some algorithmical applications, in particular the algorithm in [BCDT91] which computes the Voronoi diagram in dimension d for points constrained to belong to few hyperplanes, is generalized to power and multiplicatively weighted diagrams.

The aim of this paper is to give a simple interpretation and unify most of the results of transforming Voronoi diagrams into upper envelopes of various kind of surfaces. As it is proved by the results in [BCDT91] a better understanding of the geometric meaning of these transformations can provide new algorithms.

2 Mathematical prerequisites

We only recall here some definitions and properties, without any proof. See for example [Cox61] for more details concerning this section.

Let $\mathbb{E}^d$ be the d dimensional euclidean space and $\langle .,. \rangle$ the scalar product in $\mathbb{E}^d$.

2.1 Circles and spheres

We assume all the equations of the spheres to be given in an orthonormal basis, and to be normalized.

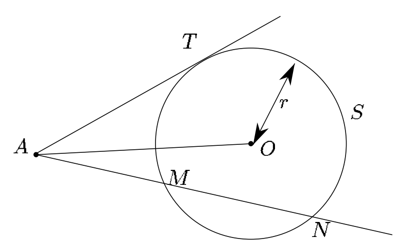

Figure 1: Power of a point with respect to a sphere

If S is a sphere with center O and radius r, and A is a point, the *power* of A with respect to S, power(A, S), is defined equivalently as (see Figure 1) :

$$\begin{aligned}
\text{power}(A,S) &= \left\langle \overrightarrow{AM}, \overrightarrow{AN} \right\rangle \\
&\quad \text{where } (AMN) \text{ is any line intersecting } S \text{ at } M \text{ and } N \\
&= \left\langle \overrightarrow{AT}, \overrightarrow{AT} \right\rangle \\
&\quad \text{if } A \text{ is exterior to } S \text{ and } (AT) \text{ is a line tangent to } S \text{ at } T \\
&= \left\langle \overrightarrow{AO}, \overrightarrow{AO} \right\rangle - r^2
\end{aligned}$$

The last definition shows that power$(A, M) > 0$ if and only if A is exterior to S.

If S has normalized equation

$$S(M) = \sum_{i=1}^{d} x_i^2 + \sum_{i=1}^{d} a_i x_i + a_0 = 0$$

we have : power$(A, S) = S(A)$. This implies the following relation which will be used in the sequel :

$$\sum_{i=1}^{k} \frac{\text{power}(A, S_i)}{w(S_i)} = \left(\sum_{i=1}^{k} \frac{1}{w(S_i)} \right) \text{power} \left(A, \frac{1}{\sum_{i=1}^{k} \frac{1}{w(S_i)}} \sum_{i=1}^{k} \frac{S_i}{w(S_i)} \right)$$

where we denote as $\sum_{i=1}^{k} \lambda_i S_i$ the sphere having as equation the corresponding linear combination of the equations of spheres $\{S_i,\ i = 1, \ldots, k\}$, and as $w(S_i)$ a weight associated to each sphere S_i.

Let us give another useful property : $\text{power}(A, S)$ is the square radius of the sphere centered on A and orthogonal to S.

The locus of a point that has the same power with respect to 2 spheres S_1 and S_2 is an hyperplane called the *chordale* of the spheres and denoted as $\Delta(S_1, S_2)$ (in the plane, it is a line called the *radical axis*). The equation of $\Delta(S_1, S_2)$ is obtained by substracting the two equations of S_1 and S_2.

A *pencil* of spheres is a linear family of spheres, i.e. the equations of the spheres of the pencil are linearly dependant of a parameter. All pairs of spheres in the pencil have the same chordale. A pencil of spheres has three equivalent definitions : it is a linear family of spheres generated by two given spheres, or the set of spheres that are orthogonal to d given spheres ; if S_1 and S_2 are two given spheres, the pencil defined by S_1 and S_2 is also the set of spheres that have the same chordale with S_1 than S_2. The different kinds of pencils will be detailed later.

2.2 Polarity

Four points M, N, A, B lying on a common line are said to form an *harmonic division* if

$$\frac{MA}{MB} = \frac{NA}{NB}$$

and we write in this case

$$(M, N, A, B) = \frac{\overline{MA}}{\overline{MB}} \cdot \frac{\overline{NB}}{\overline{NA}} = -1$$

We can remark that if $(M, N, A, B) = -1$, the sphere of diameter AB is orthogonal to any sphere passing through M and N (and symmetrically, the sphere of diameter MN is orthogonal to any sphere passing through A and B) (see Figure 2).

M and N are said to be *conjugate* with respect to a quadric Q if $(M, N, A, B) = -1$ where $\{A, B\} = (MN) \cap Q$ (see Figure 3). If M is exterior to Q, and if we choose for the line (MN) a tangent from M to Q, then $A = B = N$ is the tangent point. The locus of the conjugate of M with respect to Q is a hyperplane P passing through these tangent points.

In the projective space associated to $\mathbb{E}^d$, the projective quadric associated to Q is the kernel of a quadratic form q. In fact, the conjugation with respect to Q is nothing else that the orthogonality defined in the projective space by q ; thus the polar hyperplane of M is the set of objects orthogonal to M for the quadratic form q.

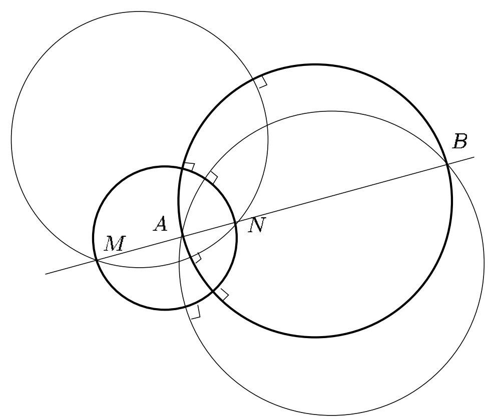

Figure 2: Harmonic division

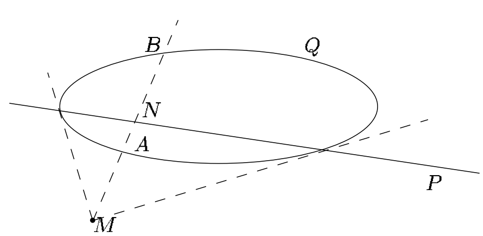

Figure 3: Conjugate points and polar hyperplane

If the equation of Q is :

$$\sum_{1 \le i \le j \le d} a_{ij} x_i x_j + \sum_{i=1}^{d} a_{0i} x_i + a_{00} = 0$$

the equation of the polar hyperplane P of M with respect to Q can be obtained by *polarizing* in M the equation of Q :

$$\sum_{i=1}^{d} a_{ii} m_i x_i + \sum_{1 \le i < j \le d} \frac{a_{ij}}{2}(m_i x_j + m_j x_i) + \sum_{i=1}^{d} \frac{a_{0i}}{2}(x_i + m_i) + a_{00} = 0$$

where $M = (m_1, \ldots, m_d)$.

3 Space of spheres

We denote by $\mathcal{P}$ the euclidean d dimensional space. $\langle .,. \rangle$ is the scalar product in $\mathcal{P}$, and $|..|$ is the euclidean distance between points. A point or a vector $(x_1, \ldots, x_i, \ldots, x_d)$ of $\mathcal{P}$ will be represented as x.

Let us recall the definition of the *space of spheres* σ used in [BCDT91]. Let

$$(S) \qquad \langle x, x \rangle - 2 \langle x, \Phi \rangle + \chi = 0 \tag{1}$$

be the equation of a sphere S in $\mathcal{P}$. Here Φ is a point of $\mathcal{P}$, namely the center of S. This sphere is represented by the point $S = (\Phi, \chi)$ in the $(d+1)$ dimensional space σ. Notice that the vertical projection of a point $S = (\Phi, \chi)$ in σ is the center of the sphere, and the radius r_S of S is given by :

$$\text{power}(O, S) = \chi = \langle \Phi, \Phi \rangle - r_S^2 \tag{2}$$

We present in this section the first properties of σ. In the sequel, we will precise the dimension of a geometric object by the following convention : a p-object is a manifold of dimension p, for example the hyperplanes of $\mathcal{P}$ will be called $(d-1)$-hyperplanes, while hyperplanes of σ will be called d-hyperplanes. There is only one exception to this convention : if no precision, the word sphere will represent the $(d-1)$-spheres in $\mathcal{P}$, since they are the usual spheres we deal with.

3.1 The paraboloid Π

Equation (2) shows that the *point-spheres* in σ (spheres of radius 0) verify :

$$(\Pi) \qquad \langle \Phi, \Phi \rangle - \chi = 0 \tag{3}$$

The set of point-spheres is in σ the unit d-paraboloid of revolution with vertical axis (that is, parallel to the χ-axis) denoted as Π.

A point (Φ, χ) on Π represents a sphere of $\mathcal{P}$ with center Φ and radius 0. Therefore points of $\mathcal{P}$ are associated with the corresponding point-spheres on Π in σ. If space $\mathcal{P}$

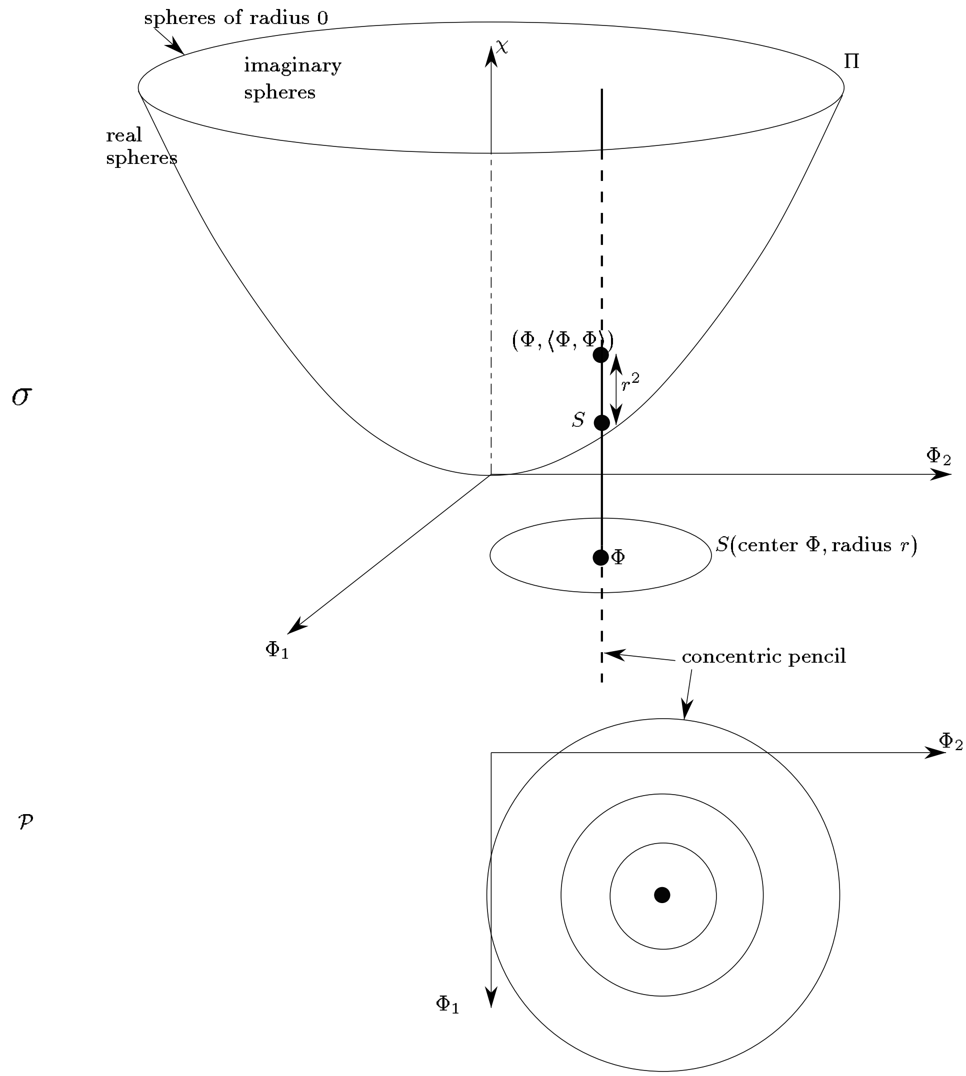

Figure 4: The space of spheres

is identified to the d-hyperplane $\chi = 0$ then the point-sphere is obtained by raising the point on Π. The exterior of Π is the set of real spheres, whereas its interior is the set of *imaginary* spheres (with negative square radius) (see Figure 4).

The square radius of a sphere $S = (\Phi, \chi)$ is the difference of the χ-coordinate of the vertical projection of S on Π with χ.

3.2 Hyperplanes in σ

Two spheres of $\mathcal{P}$ are orthogonal if and only if the two corresponding points of σ are conjugate with respect to Π. Thus the set of spheres of $\mathcal{P}$ orthogonal to a given sphere $S_0 = (\Phi_0, \chi_0)$ of $\mathcal{P}$ forms in σ exactly the polar d-hyperplane π_{S_0} of point S_0 with respect to Π. Its equation is obtained by polarizing the equation of Π in S_0 :

$$(\pi_{S_0}) \qquad \chi = 2\langle \Phi_0, \Phi \rangle - \chi_0 \tag{4}$$

It is the polar d-hyperplane of the sphere with respect to Π.

The intersection of π_{S_0} and Π is precisely the set of point-spheres that are orthogonal to sphere S_0, that is the set of points lying on S_0 in $\mathcal{P}$. This shows that $\pi_{S_0} \cap \Pi$ projects in $\mathcal{P}$ onto S_0 (see Figure 5).

As a particular case, the set of spheres passing through a point $M \in \mathcal{P}$ is also the set of spheres orthogonal to the point-sphere M. But, as M belongs to Π, the polar d-hyperplane π_M is nothing else than the tangent d-hyperplane to Π at M. Each sphere in the lower half space limited by π_M (i.e. the half space which does not contain Π), contains M in its interior. For a sphere in the upper half space, M is outside (see Figure 5).

3.3 Lines in σ

A pencil of spheres, that is the set of linear combinations of two spheres of $\mathcal{P}$, transforms in σ into the line through the two corresponding points.

From Equation (2), a concentric pencil is a vertical line (see Figure 4). More generally, a pencil of spheres with limit points is a line hitting Π in the two limit point-spheres (see Figure 6).

A pencil of spheres with base points has d base points forming a $(d-1)$-simplex. All the spheres in the pencil intersect along the $(d-2)$-sphere circumscribing the $(d-1)$-simplex. Such a pencil is a line which does not hit Π, this line is the intersection of the d polar d-hyperplanes of the base points (see Figure 7).

A pencil with tangent point Φ is a line tangent to Π at the projection of Φ on Π (see Figure 8).

More generally, if S_1 and S_2 are two spheres, the intersection of their polar d-hyperplanes π_{S_1} and π_{S_2} is the set of all spheres orthogonal to both S_1 and S_2. All these spheres are necessarily centered on the chordale $\Delta(S_1, S_2)$. Thus $\pi_{S_1} \cap \pi_{S_2}$ projects on $\Delta(S_1, S_2)$ (see Figure 9).

The points at infinity of the projective closure of σ correspond to the $(d-1)$-hyperplanes of $\mathcal{P}$, namely the point at infinity in the direction of (Φ, χ) is the $(d-1)$-hyperplane of equation $-2\langle \Phi, x \rangle + \chi = 0$, and the point at infinity of a line in σ is the

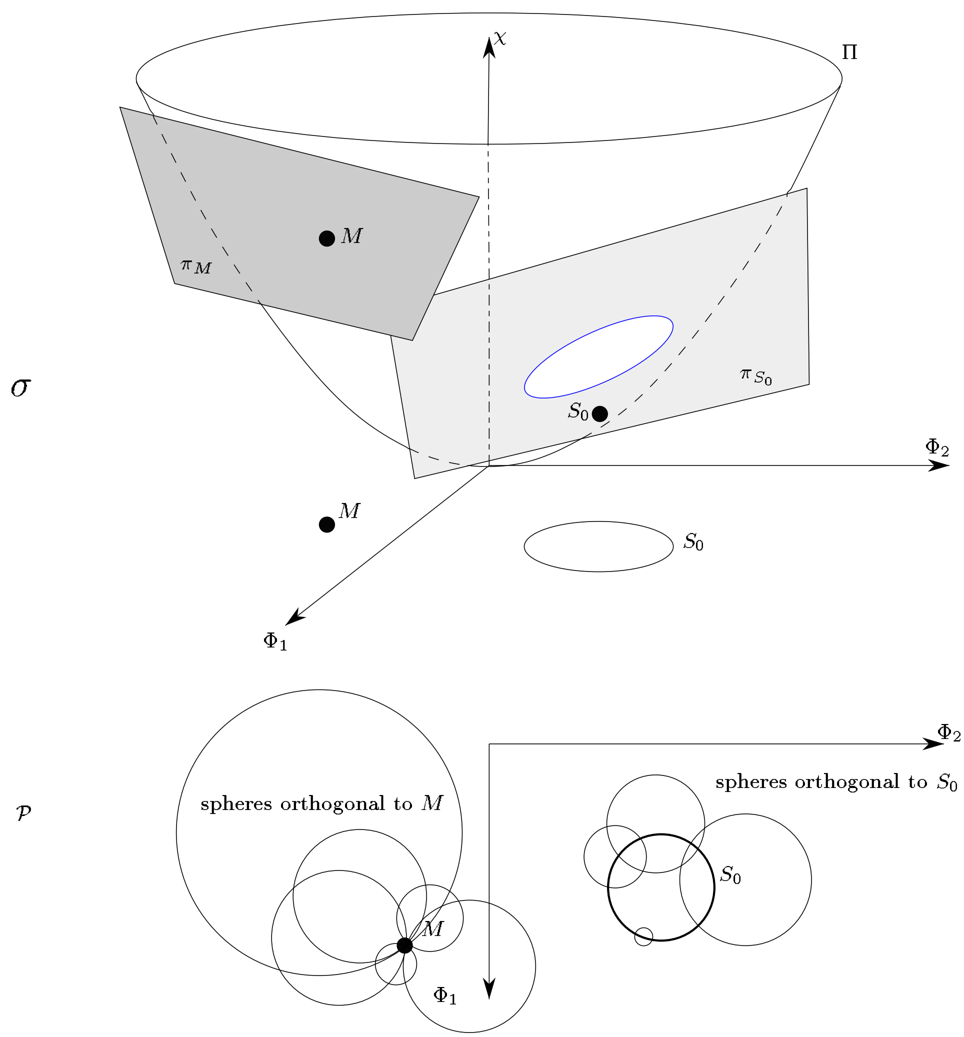

Figure 5: The polar hyperplane of a sphere

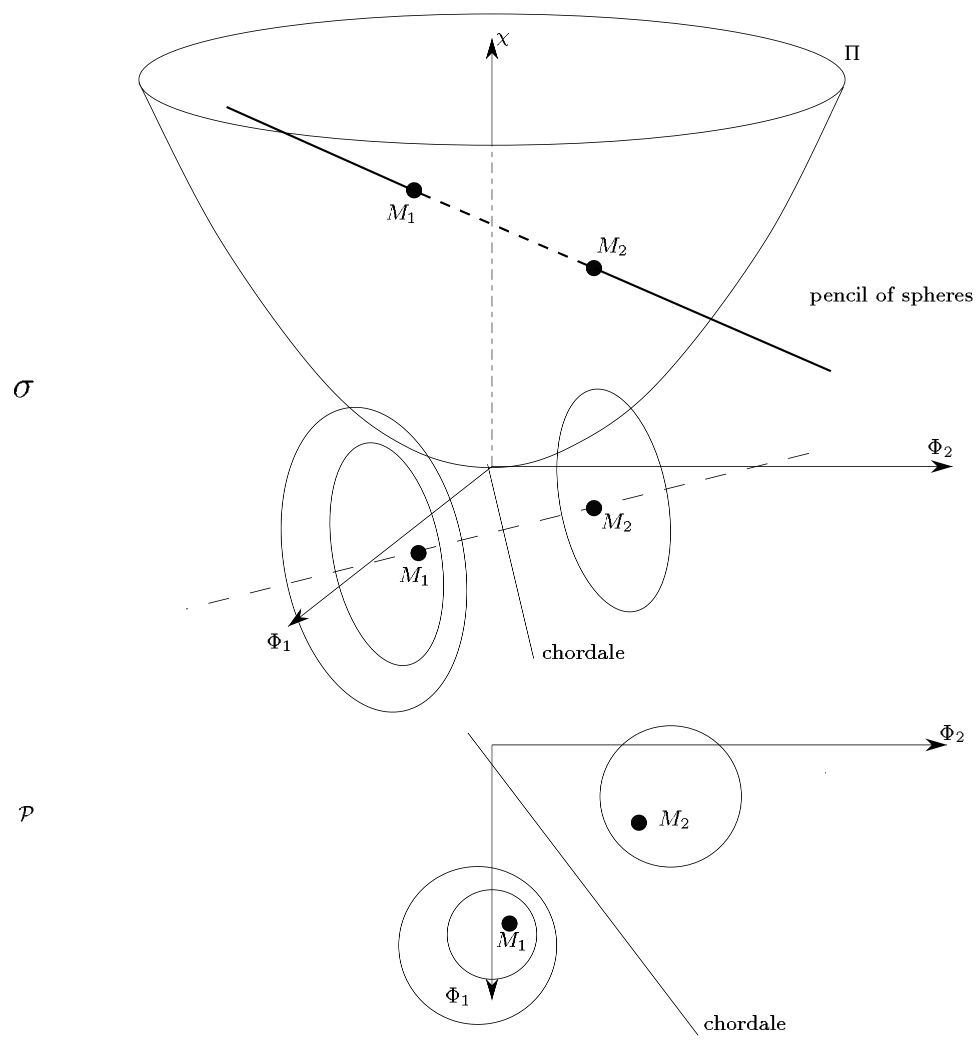

Figure 6: A pencil of spheres with limit points

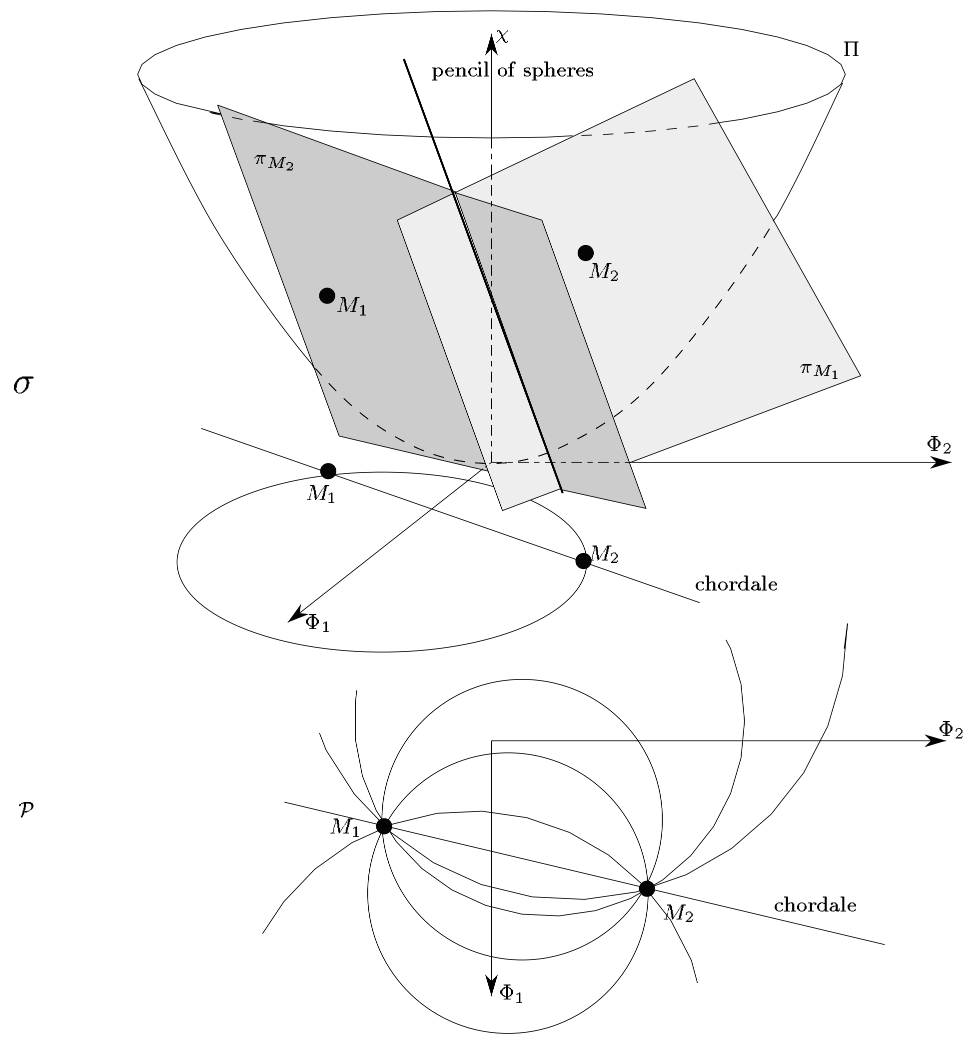

Figure 7: A pencil of spheres with base points

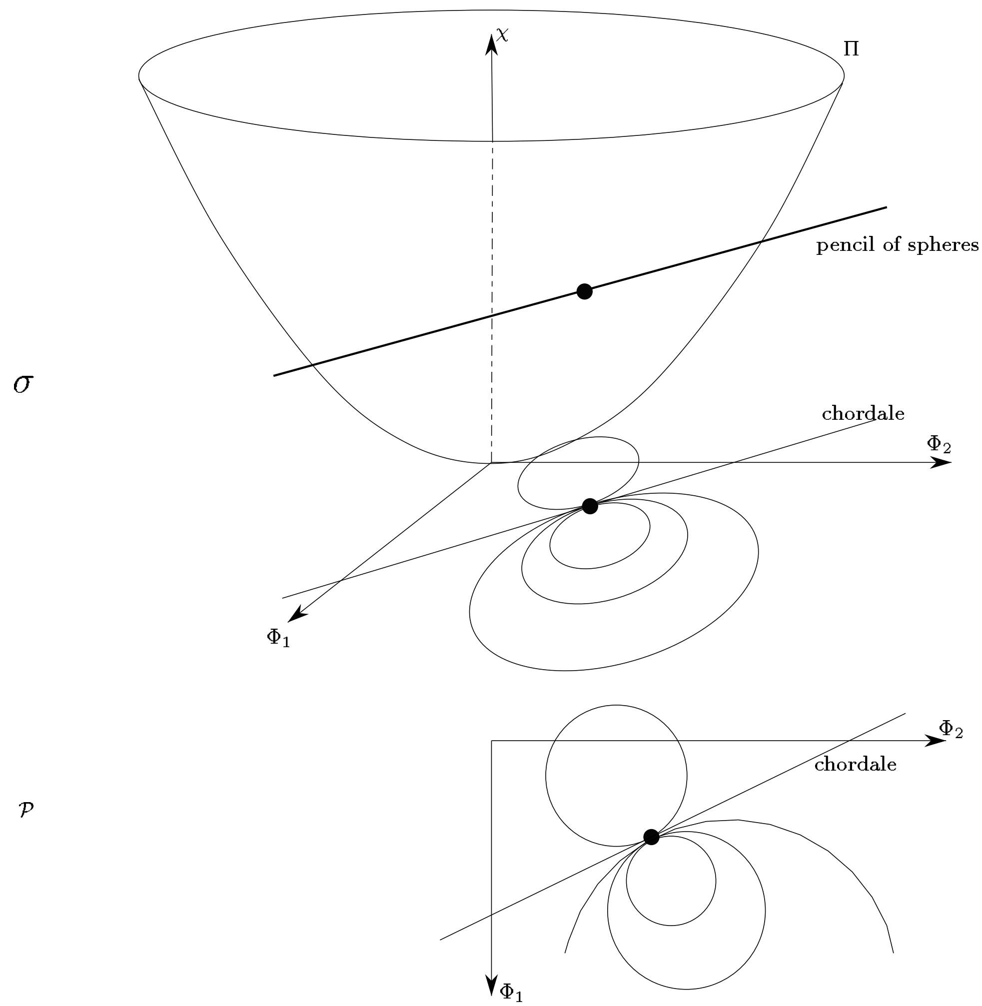

Figure 8: A pencil of spheres with tangent point

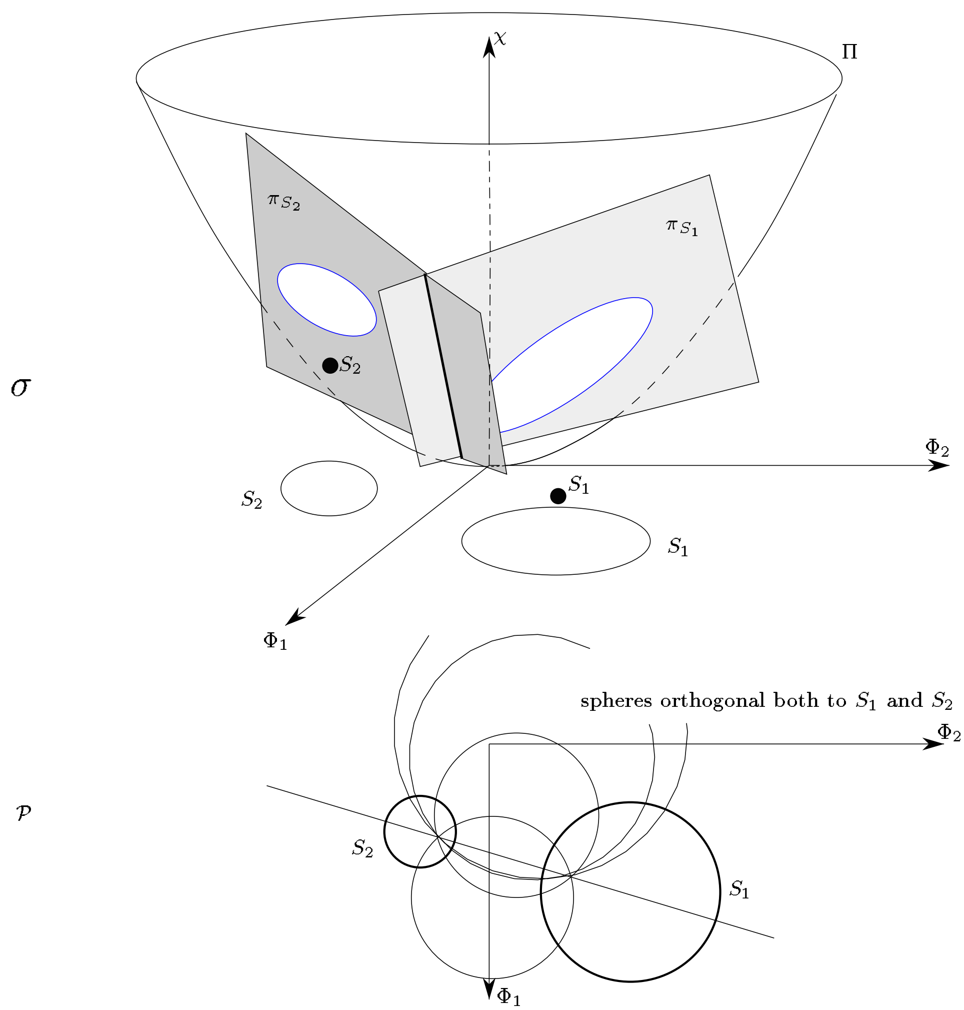

Figure 9: $\pi_{S_1} \cap \pi_{S_2}$

chordale of the corresponding pencil. Thus the pencils whose chordale is a given $(d-1)$-hyperplane in $\mathcal{P}$ form in σ the set of lines parallel to a given direction (orthogonal to the chordale). In particular, a pencil of spheres having in $\mathcal{P}$ the $(d-1)$-hyperplane of equation $x_d = 0$ as its chordale corresponds in σ to a line parallel to the Φ_d-axis.

3.4 The map ρ_k

We denote by ρ_k the transformation in σ that maps a sphere $S = (\Phi, \chi)$ with sqare radius r^2 in σ to a sphere $\rho_k(S)$ having the same center and square radius kr^2, for $k \in I\!R$.

Since the two centers are equal, S and $\rho_k(S)$ are vertically aligned in σ. Let us call χ_k the last coordinate of $\rho_k(S)$. From the beginning of Section 3, we know that the χ-coordinate of a sphere is the power of the origin with respect to the sphere. So :

$$\begin{array}{rrcl} & \chi & = & \langle \Phi, \Phi \rangle - r^2 \\ \text{and} & \chi_k & = & \langle \Phi, \Phi \rangle - kr^2 \\ \text{thus} & \chi_k & = & (1-k)\langle \Phi, \Phi \rangle + k\chi \end{array}$$

By using the preceding equation, together with Equation (4), we can immediately see that the polar d-hyperplane π_{S_0} for a sphere $S_0 = (\Phi_0, \chi_0)$ in σ, maps through ρ_k to the d-paraboloid $\rho_k(\pi_{S_0})$ of equation

$$(\rho_k(\pi_{S_0})) \qquad \chi = (1-k)\langle \Phi, \Phi \rangle + k(2\langle \Phi_0, \Phi \rangle - \chi_0) \tag{5}$$

The axis of this d-paraboloid is vertical (see Figure 10). Since the restriction of ρ_k to Π is evidently the identity, we have :

$$\rho_k(\pi_{S_0}) \cap \Pi = \pi_{S_0} \cap \Pi$$

This intersection is, if S_0 is a real sphere (with positive square radius), the vertical projection of S_0 onto Π (see Section 3.2).

As a particular case, if S_0 is a point-sphere, π_{S_0} is the d-hyperplane tangent to Π at point $(\Phi_0, \langle \Phi_0, \Phi_0 \rangle)$, and $\rho_k(\pi_{S_0})$ is also tangent to Π at the same point.

4 Duality results

In the following sections, we apply our framework to different kinds of Voronoi diagrams. The resulting transformation is often already well known, however, using our framework, the transformation is straightforward and does not require any calculus. Furthermore, our framework allows an easy combination of all possible generalizations of Voronoi diagrams, such as weighted order k power diagrams. It may also lead to new efficient algorithms (see Section 5).

The terms such as distance, or nearest neighbor, refer in each section to the specific distance δ defined in that particular section.

In the sequel, $\mathcal{S}$ always denotes a set of spheres. An element of $\mathcal{S}$ is denoted by S or S_i. In some cases, $\mathcal{S}$ is restricted to a special class of spheres, e.g. point-spheres in Section 4.1.

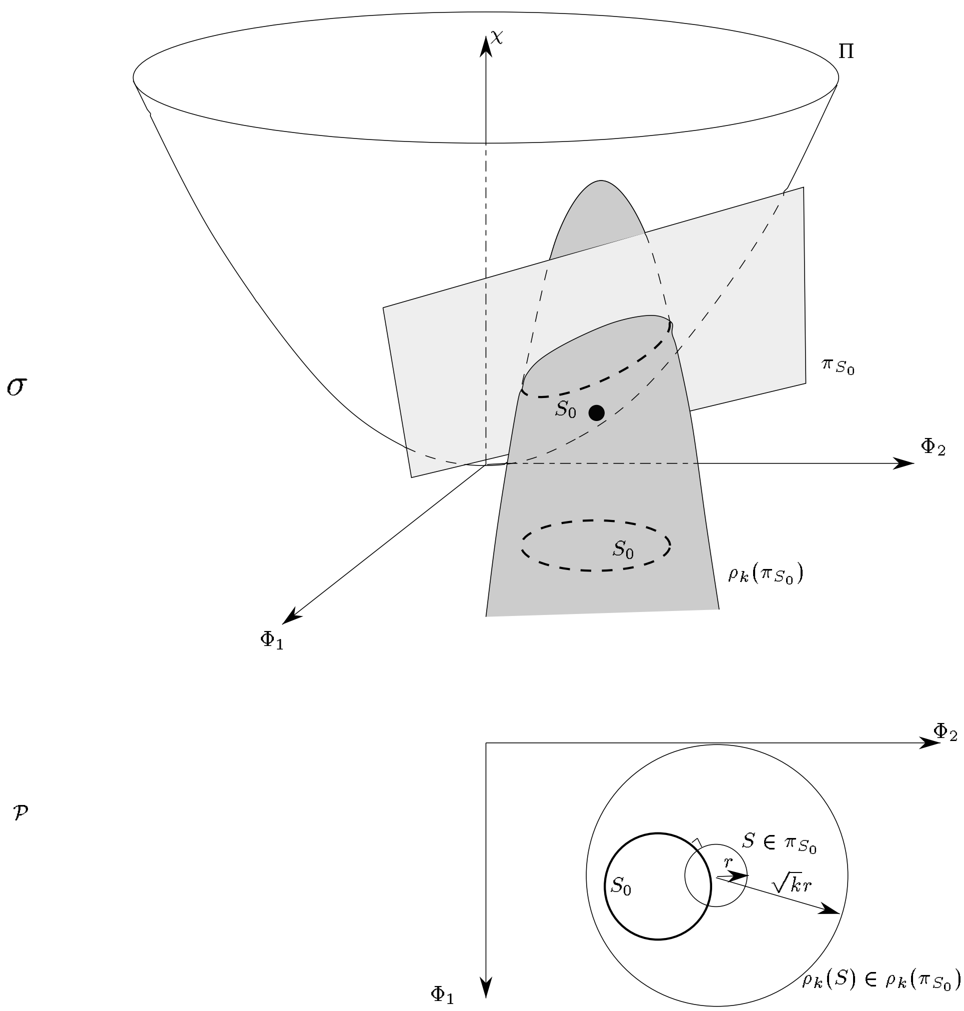

Figure 10: The map ρ_k

4.1 Voronoi diagrams of point sites

The distance we consider here is the euclidean distance in $\mathcal{P}$:

$$\delta(P, Q) = |PQ|$$

As noticed in Section 3.2, the set of spheres which do not contain a given point maps in σ into the upper half space, limited by the polar d-hyperplane of the corresponding point. Thus, if $\mathcal{S}$ is a set of sites in space $\mathcal{P}$, then the set of empty spheres (i.e. the set of spheres which do not surround any site of $\mathcal{S}$) of space $\mathcal{P}$ is in σ the intersection of the corresponding half spaces. Its boundary is a convex polytope $U_{\mathcal{S}}$, whose facets are tangent to Π.

For a given point $M \in \mathcal{P}$ and a site $S \in \mathcal{S}$, the intersection of the vertical line through M with π_S gives the maximum empty sphere centered at M and touching S (see Figure 11). Since the radius of the spheres on the vertical line increases with falling χ, we have :

> *$\mathcal{P}$ 1 The intersection of $U_{\mathcal{S}}$ with a vertical line $\Phi = M$ in σ gives the sphere with center M and whose radius is the distance to the nearest neighbor of M in $\mathcal{S}$.*

In other words the projection of $U_{\mathcal{S}}$ on $\mathcal{P}$ is the Voronoi diagram of $\mathcal{S}$. This can be viewed as a new presentation of the well known correspondence between intersection of half spaces tangent to the d-paraboloid and Voronoi diagram [ES86].

This first correspondence in the space of spheres has been used in [BCDT91] to compute the 3D Delaunay triangulation of a set of sites lying in k different planes, with an output sensitive complexity (see Section 5.2.1).

The space of spheres allows more general statements : the intersection of $U_{\mathcal{S}}$ with a line representing any pencil of spheres, is made up of the extremal empty spheres of this pencil. There are zero, one, or two such extremal spheres (unless it is a pencil contained in the polar plane of a site). These spheres are empty spheres passing through a site of $\mathcal{S}$. Because there is generally exactly one sphere of a pencil passing through a given site, the determination of the extremal empty spheres of a given pencil with respect to $\mathcal{S}$ can be viewed equivalently, as the determination of the points of $\mathcal{S}$ on these spheres.

4.2 Power diagrams

The power diagram is defined for a set $\mathcal{S}$ of spheres of $\mathcal{P}$. For a sphere S and a point M in $\mathcal{P}$, we define the distance $\delta(M, S)$ from M to S as the power of M with respect to S. This distance is also called Laguerre distance [Aur91].

$$\delta(M, S) = \text{power}(M, S)$$

In this definition, the spheres of $\mathcal{S}$ can be any spheres, even point-spheres or imaginary spheres. In the case the spheres reduce to points, δ is the squared usual euclidean distance, and we obtain the usual Voronoi diagram for point sites as a particular case of power diagram.

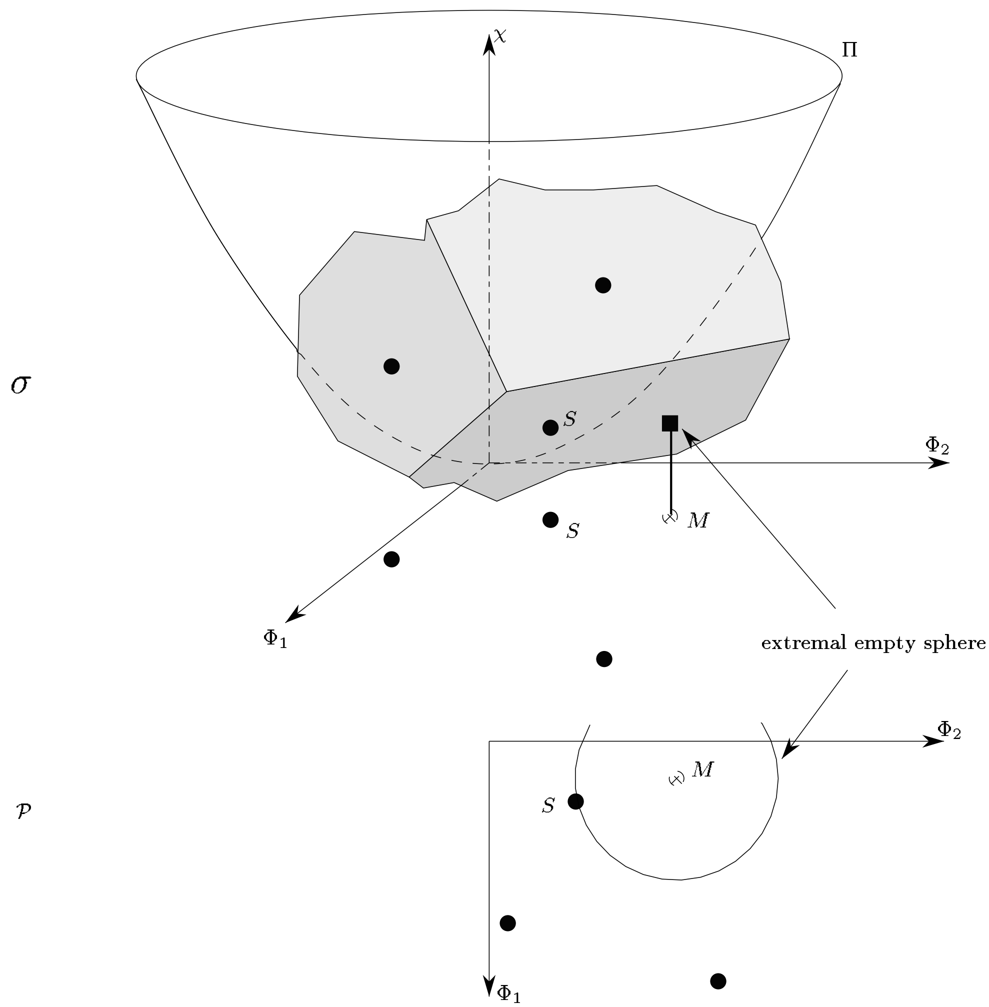

Figure 11: Empty sphere centered on M and touching S

In space $\mathcal{P}$, the power of M with respect to S is the square distance from M to the intersection point with S of a $(d-1)$-hyperplane tangent to S and passing through M. It can also be seen as the square radius of the sphere which is both centered on M and is orthogonal to S. In σ, this sphere is the intersection of the vertical line passing through M and the polar d-hyperplane π_S. If we now consider all spheres in $\mathcal{S}$, the upper envelope of all their polar d-hyperplanes, or equivalently, the intersection of the upper half spaces limited by these d-hyperplanes, form a convex polytope $U_{\mathcal{S}}$, and the closest sphere to M for distance δ is given by the intersection of $U_{\mathcal{S}}$ with the vertical line through M.

> $\mathcal{P}$ *2 The intersection of* $U_{\mathcal{S}}$ *with a vertical line* $\Phi = M$ *in* σ *gives the sphere with center* M *and whose square radius is the distance from* M *to its nearest neighbor in* $\mathcal{S}$.

This shows that the power diagram of the set of spheres $\mathcal{S}$ can be obtained by projecting onto $\mathcal{P}$ the upper envelope of the polar d-hyperplanes of the spheres.

The same result has been proved in [Aur87] in a different way, without any geometric interpretation.

In the power diagram, a $(d-1)$-face separating the regions of two spheres is contained in the projection of the intersection of the two corresponding polar d-hyperplanes. This also follows from the statement in Section 3.3 that the intersection of two polar d-hyperplanes projects to the chordale of the corresponding spheres.

4.3 Weighted power diagrams

The sites of $\mathcal{S}$ can here be any kind of spheres. Each site S is assigned a weight $w(S)$. The weighted distance of a point to a site is now defined as the power divided by the weight of the site.

$$\delta(M, S) = \frac{\text{power}(M, S)}{w(S)}$$

Let S denote a site in $\mathcal{S}$ ans let $w(S)$ be its weight. Consider a sphere $S_M = (M, \chi)$, centered on M on the paraboloid $\rho_{\frac{1}{w(S)}}(\pi_S)$. $S_M = \rho_{\frac{1}{w(S)}}(S'_M)$ for some sphere $S'_M \in \pi_S$ whose square radius is equal to power(M, S). So the square radius of S_M is $\frac{\text{power}(S,M)}{w(M)} = \delta(M, S)$.

Let us now define $U_{\mathcal{S}}$ as the upper envelope of the set of d-paraboloids $\{\rho_{\frac{1}{w(S)}}(\pi_S), S \in \mathcal{S}\}$. Intersecting a vertical line through M with $U_{\mathcal{S}}$ gives the site S which minimizes $\delta(M, S)$.

> $\mathcal{P}$ *3 The intersection of* $U_{\mathcal{S}}$ *with a vertical line* $\Phi = M$ *in* σ *gives the sphere with center* M *and whose square radius is the distance from* M *to its nearest neighbor in* $\mathcal{S}$.

The weighted power diagram of $\mathcal{S}$ is thus the projection on $\mathcal{P}$ of the upper envelope of the set of d-paraboloids $\{\rho_{\frac{1}{w(P)}}(\pi_P), P \in \mathcal{S}\}$.

The first algorithm for computing weighted Voronoi diagrams for point sites in the plane is due to F. Aurenhammer and H. Edelsbrunner [AE84], and uses a duality that is somewhat different from ours. In [Aur87], there is also a duality with the upper envelope of a set of d-paraboloids for the computation of weighted Voronoi diagrams, but the d-paraboloids are not the same as ours.

Let us finaly remark the following : the weighted distance $dist(M,S)$ between a point M and a point site $S \in \mathcal{S}$ is usually defined to be the euclidean distance divided by the weight $weight(S)$ of the site. Therefore we have :

$$dist^2(M,S) = \frac{|MS|^2}{\mathrm{weight}^2(S)} = \frac{\mathrm{power}(M,S)}{\mathrm{weight}^2(S)} = \delta(M,S)$$

The weighted Voronoi diagram for point sites can therefore be obtained from the power diagram of point-spheres with squared weights.

4.4 Affine Voronoi diagrams

By *affine* Voronoi diagram, we denote abstract Voronoi diagrams whose bisectors are hyperplanes. F. Aurenhammer has shown that every affine Voronoi diagram is a power diagram. More precisely, he proved the following theorem :

Theorem [Aurenhammer] *If $\{H_{ij}, 1 \leq i < j \leq n\}$ is a set of hyperplanes in an euclidean space, verifying the condition $\forall i < j < k, H_{ij} \cap H_{ik} = H_{ij} \cap H_{jk} = H_{ik} \cap H_{jk}$, then the Voronoi diagram in which the bisectors are the hyperplanes $\{H_{ij}, 1 \leq i < j \leq n\}$ is the power diagram of a set of spheres.*

Proof. See [Aur87]. F. Aurenhammer gives the construction of a set of spheres. This set is not unique. One of these spheres can be chosen arbitrarily, the other spheres are computed so that each chordale of two spheres is an hyperplane H_{ij}. □

In [Aur87], this result is applied to deduce a duality between weighted Voronoi diagrams for point sites in dimension d and power diagrams in dimension $d+1$. In that way, weighted diagrams reduce to non-weighted diagrams. We give an example in Section 5.2.3, showing that this transformation can nevertheless increase the complexity of the problem.

Here, we can show a duality between the weighted diagram in $\mathcal{P}$ and a power diagram in σ. For P and Q two spheres in $\mathcal{P}$, we define the d-bisector of P and Q as the d-hyperplane H_{PQ} of σ containing the intersection of the two d-paraboloids $\rho_{\frac{1}{w(P)}}(\pi_P)$ and $\rho_{\frac{1}{w(Q)}}(\pi_Q)$. Note that this intersection cannot be empty and forms a d-ellipsoid or a d-paraboloid. We must determine the d-spheres σ_P and σ_Q in σ (they are d-spheres in the space of spheres) associated respectively with P and Q such that their d-chordale $\Delta(\sigma_P, \sigma_Q)$ is H_{PQ}.

From Equation 5, the normalized equation of $\rho_{\frac{1}{w(P)}}(\pi_P)$ is

$$(\rho_{\frac{1}{w(P)}}(\pi_P)) \qquad \langle \Phi, \Phi \rangle - \frac{2}{1-w(P)} \langle \Phi_P, \Phi \rangle + \frac{w(P)}{1-w(P)} \chi + \frac{1}{1-w(P)} \chi_P = 0$$

where (Φ_P, χ_P) are the coordinates of P in σ. The equation of H_{PQ} can be obtained by substracting $(\rho_{\frac{1}{w(P)}}(\pi_P))$ from $(\rho_{\frac{1}{w(Q)}}(\pi_Q))$.

If we add a term χ^2 in the equation of $(\rho_{\frac{1}{w(P)}}(\pi_P))$, we obtain the equation of a d-sphere σ_P in σ ; the d-chordale of two such d-spheres σ_P and σ_Q, whose equation is obtained by substraction, is H_{PQ}.

The coordinates of the center of d-sphere σ_P are :

$$(\Phi, \chi) = \left(\frac{1}{1 - w(P)}\Phi_P, \frac{w(P)}{2(w(P) - 1)}\right) \tag{6}$$

Let us remark that the last coordinate of the center does not depend on P, but only on its weight. This remark will be used in the algorithm of Section 5.2.3.

When the power diagram of the d-spheres $\{\sigma_P, P \in \mathcal{S}\}$ is computed, the weighted diagram of $\mathcal{S}$ is obtained by projection. More precisely, the region of a site P in $\mathcal{P}$ is the vertical projection of the intersection of $\rho_{\frac{1}{w(P)}}(\pi_P)$ with the cell of σ_P in the power diagram in σ(see [Aur87]).

A triangulation can be viewed as an affine diagram, thus, from the above theorem, it is a power diagram. It can easily be seen that it is the power diagram of the spheres circumscribing the simplices.

4.5 Order k power diagrams

The sites of $\mathcal{S}$ can be any spheres, and the distance is :

$$\delta(M, S) = \text{power}(M, S)$$

In the order k power diagram, the regions are associated to subsets of $\mathcal{S}$ with cardinality k. Let T be such a subset. The power region of T is the locus of a point whose k nearest neighbors are the elements of T.

F. Aurenhammer and H. Imai [AI88,Aur90] use a duality for the computation of these diagrams. Our interpretation allows to give an easy proof. The following relation, for k elements $S_1, \ldots, S_k \in \mathcal{S}$ has been given in Section 2 :

$$\frac{1}{k}\sum_{i=1}^{k} \text{power}(M, S_i) = \text{power}\left(M, \frac{1}{k}\sum_{i=1}^{k} S_i\right)$$

Let $\mathcal{S}_k$ denote the set of the centers of mass in σ of all possible k-tuples of spheres of $\mathcal{S}$. It follows that the spheres $S_1, \ldots, S_i, \ldots, S_k$ are the k nearest neighbors of M if and only if the power of M with respect to their center of mass is smaller than the power of M with respect to the other elements of $\mathcal{S}_k$.

This means exactly that the order k power diagram of a set $\mathcal{S}$ of spheres is the usual power diagram of $\mathcal{S}_k$.

Using ($\mathcal{P}$ *2*), we deduce :

> $\mathcal{P}$ *5 The intersection of* $U_{\mathcal{S}_k}$ *with a vertical line* $\Phi = M$ *in* σ *gives the sphere with center* M *and whose square radius is the average of the powers of* M *with respect to the* k *nearest neighbors of* M *in* $\mathcal{S}$.

Weighted order k power diagram

The powerful tools provided by the space of spheres allow to easily combine weighted diagrams with order k power diagrams, which could not be done in a straightforward manner with the methods previously used in the literature. We only have to use the relation

$$\sum_{i=1}^{k} \frac{\text{power}(M, S_i)}{w(S_i)} = \left(\sum_{i=1}^{k} \frac{1}{w(S_i)}\right) \text{power}\left(M, \frac{1}{\sum_{i=1}^{k} \frac{1}{w(S_i)}} \sum_{i=1}^{k} \frac{S_i}{w(S_i)}\right)$$

and we can see that it implies that the weighted order k Voronoi diagram of a set of sites is the projection of the upper envelope of the d-paraboloids obtained, for each k-tuple $\{S_1,\ldots,S_i,\ldots,S_k\}$, by applying $\rho_{\left(\frac{1}{\sum_{i=1}^{k} \frac{1}{w(S_i)}}\right)}$ to the polar d-hyperplane π_C of the centroid $C_{S_1,\ldots,S_i,\ldots,S_k} = \frac{1}{\sum_{i=1}^{k} \frac{1}{w(S_i)}} \sum_{i=1}^{k} \frac{S_i}{w(S_i)}$ of the points $S_1, \ldots, S_i, \ldots, S_k$ in σ with respective coefficients $\frac{1}{w(S_i)}$.

Let $\mathcal{S}_k$ be the set of these centroids $C_{S_1,\ldots,S_i,\ldots,S_k}$, with weight $\sum_{i=1}^{k} \frac{1}{w(S_i)}$, for all k-tuples $\{S_1,\ldots,S_i,\ldots,S_k\}$ of sites of $\mathcal{S}$.

The weighted order k power diagram of $\mathcal{S}$ is the (order 1) weighted power diagram of $\mathcal{S}_k$.

4.6 Voronoi diagrams of general manifolds

We can now study some very general problems. The elements of $\mathcal{S}$ are now manifolds of any dimensions immersed in $\mathcal{P}$. The distance is the usual distance from a point M to a manifold Z ($|MP|$ is the euclidean distance between M and P) :

$$\delta(M, Z) = \min_{P \in Z} |MP|$$

In other words, $\delta(M, Z)$ is the radius of the minimum sphere centered at M and "tangent" to Z. By *tangent* we mean that Z intersect the sphere but not its interior.

This sphere can be obtained as the intersection in σ of the vertical line $\Phi = M$ with the set Γ_Z of all spheres tangent to Z.

Γ_Z is a manifold in σ, it is the upper envelope of the polar planes of the point-spheres of Z. If Z is an analytic manifold in $\mathcal{P}$, then Γ_Z is an analytic manifold in σ. We consider the upper envelope $U_{\mathcal{S}}$ of the manifolds Γ_Z for all $Z \in \mathcal{S}$.

> $\mathcal{P}$ 6 *The intersection of $U_{\mathcal{S}}$ with a vertical line $\Phi = M$ in σ gives the sphere with center M and whose radius is the distance to the nearest neighbor of M in $\mathcal{S}$.*

Hyperplanes

If the sites of $\mathcal{S}$ are $(d-1)$-hyperplanes of $\mathcal{P}$, the vertical d-hyperplane of σ containing a site H is the set of spheres centered on H, which is the set of all spheres that are orthogonal to H, that is π_H (as in Section 3.3, H is considered as a sphere with infinite radius, or equivalently as a point at infinity of σ). The projection of H onto Π is the $(d-1)$-paraboloid $\pi_H \cap \Pi$.

Γ_H is the parabolic d-cylinder of generatrix $\pi_H \cap \Pi$ and directrix of direction H (see Figure 12).

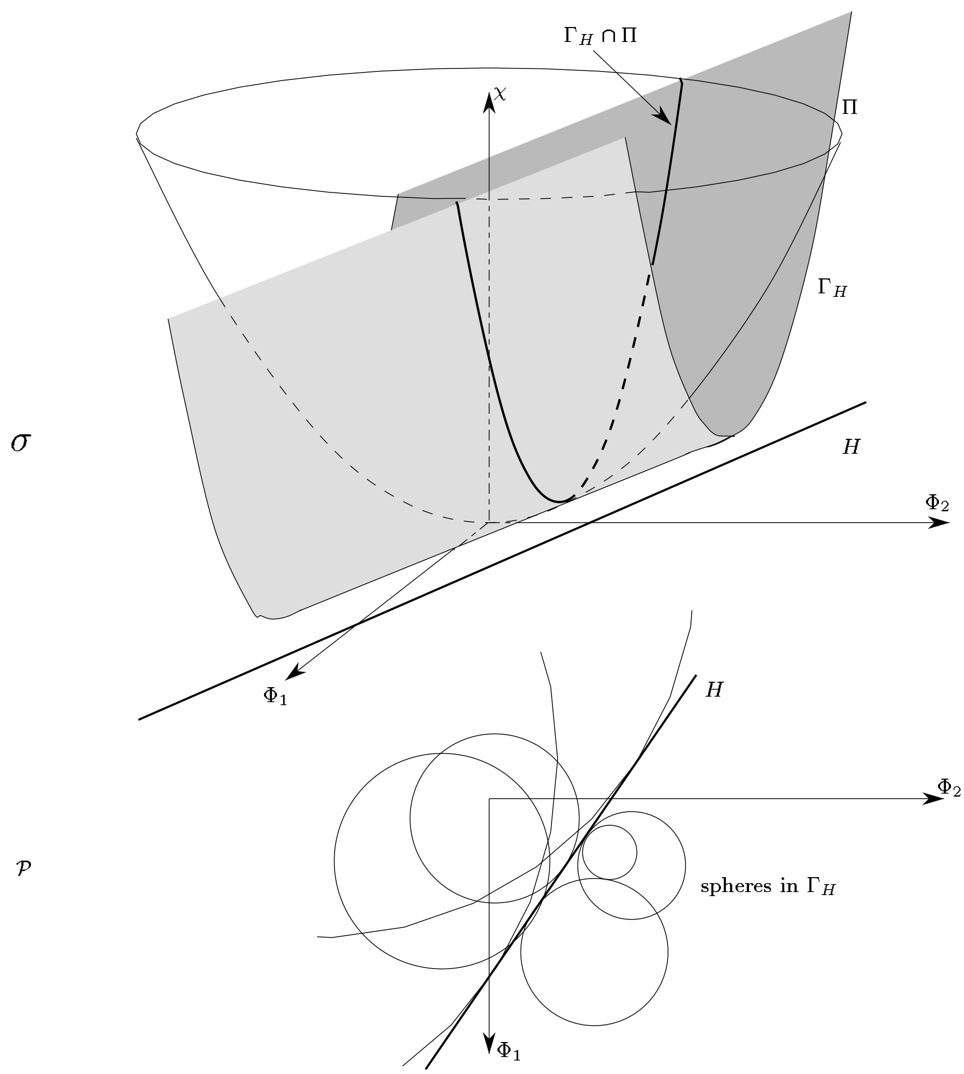

Figure 12: The manifold Γ_H for a $(d-1)$-hyperplane

The map ρ_k transforms Γ_H into a d surfece of degree 2 (a hyperbolic paraboloid when $d = 2$).

Portions of hyperplanes

If we only consider a portion of $(d-1)$-hyperplane H, the manifold Γ associated with it is still a d-cylinder with the same directrix, but its generatrix is only a portion of the $(d-1)$-paraboloid $\pi_H \cap \Pi$, extended by its two tangent d-hyperplanes (see Figure 13).

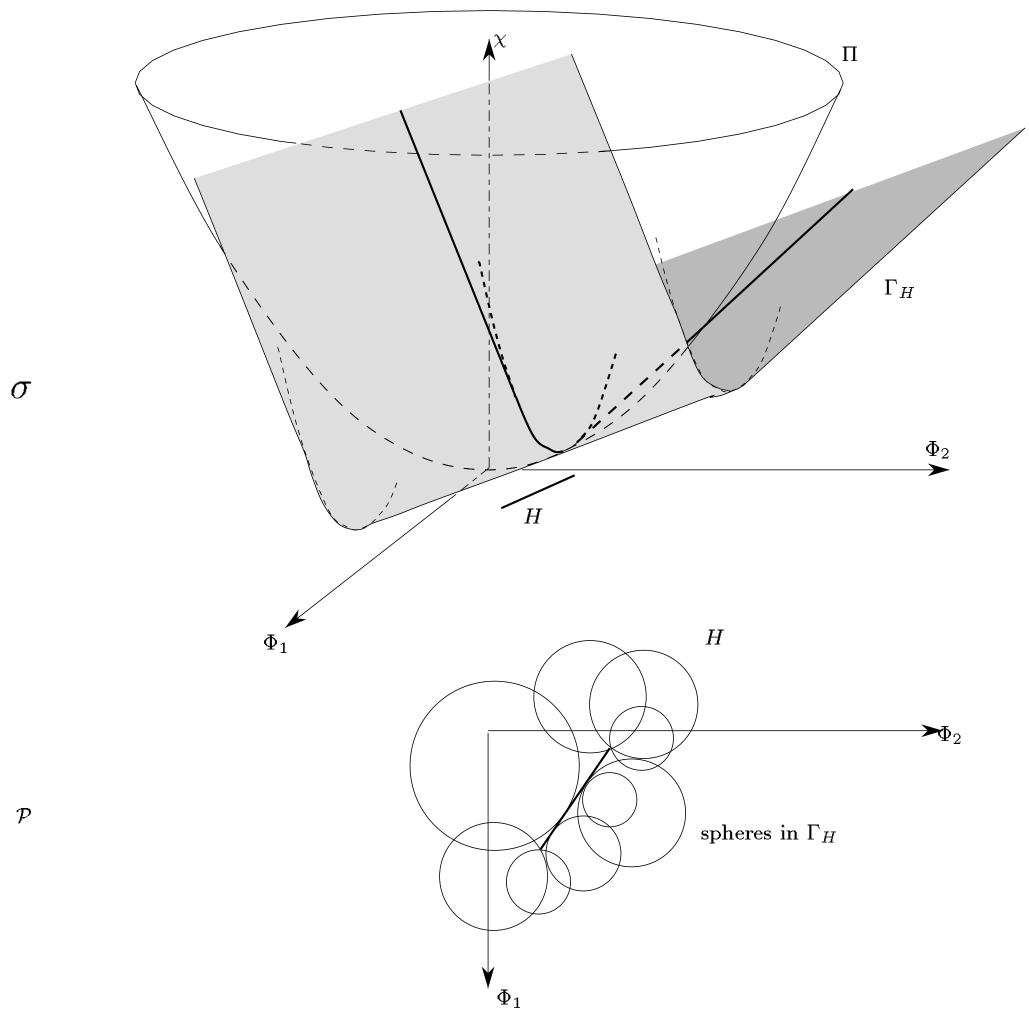

Figure 13: The manifold Γ_H for a portion of a $(d-1)$-hyperplane

Spheres

If the manifold S is a sphere, a sphere is tangent to S at point $P \in S$ if and only if it belongs to the pencil with tangent point defined by S and the point-sphere P.

Γ_S is the set of lines tangent to Π and passing through S in σ, which is also the upper envelope of the polar planes of the point-spheres lying on S in $\mathcal{P}$. It is a d-cone (see Figure 14).

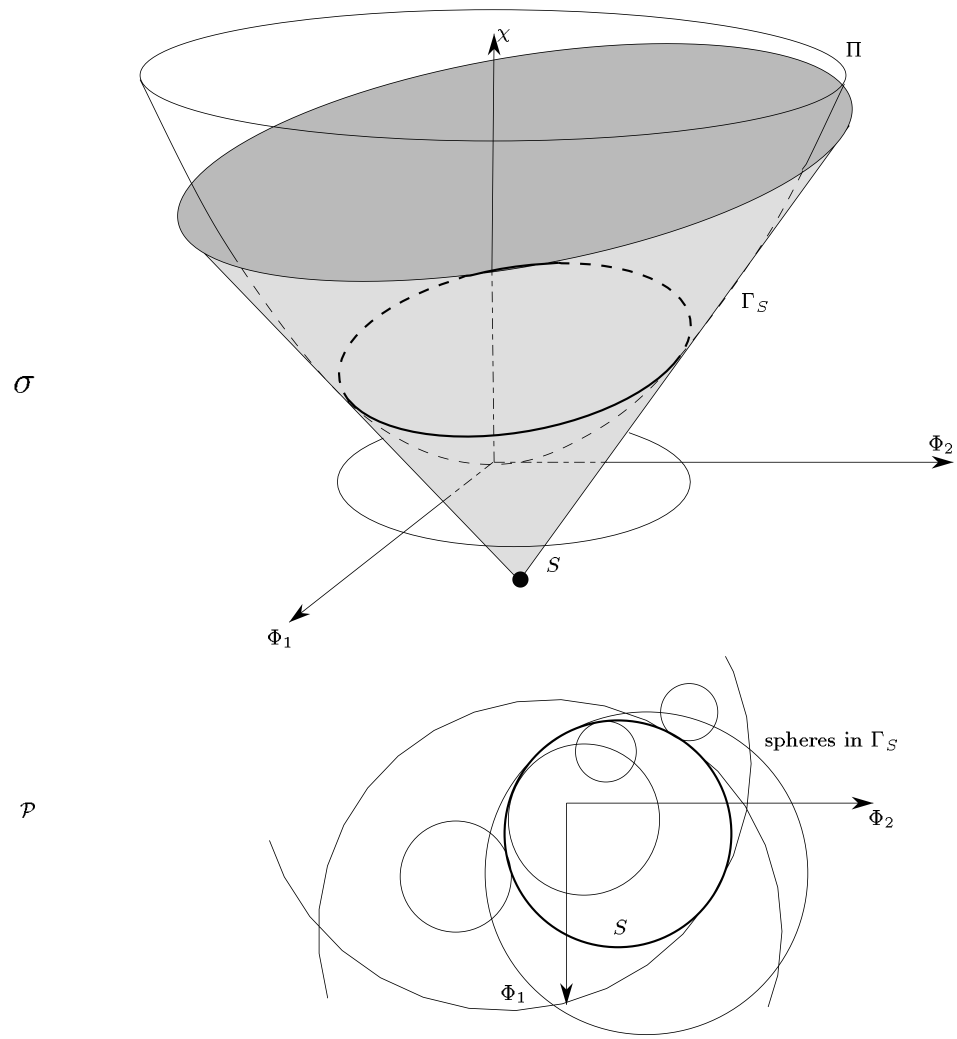

Figure 14: The manifold Γ_S for a sphere

The map ρ_k transforms Γ_S into a manifold of degree 4.

Portions of spheres

If S is a portion of a sphere, Γ_S is a portion of a d-cone extended by its two tangent d-hyperplanes (see Figure 15).

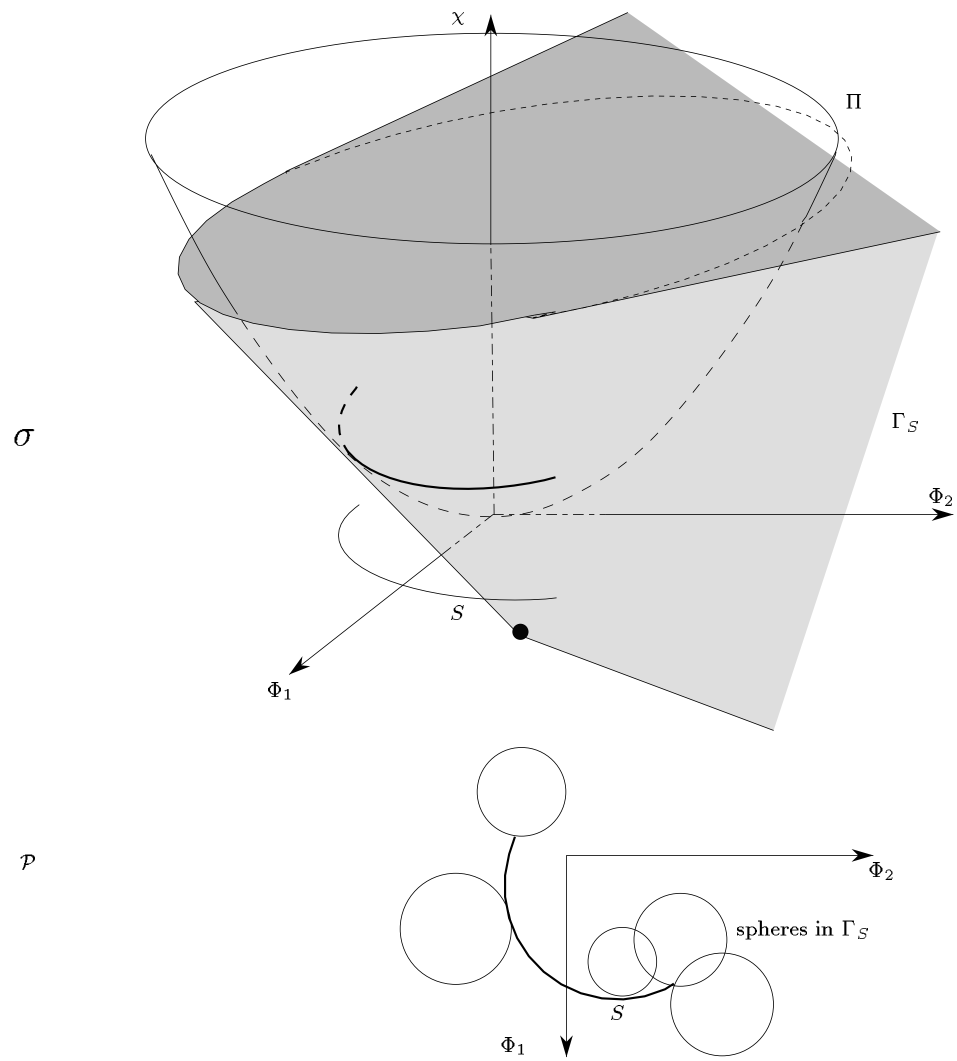

Figure 15: The manifold Γ_S for a portion of a sphere

4.7 Hyperbolic Voronoi diagrams

This section presents an application of the space of spheres in the context of (non euclidian) hyperbolic geometry (a good introduction can be found in [Bea83]). The most classical conformal model of the 2 dimensional non euclidian space (of Lobachevsky) is the Poincaré upper half plane $I\!P = \{x + iy \in \mathbb{C} \,|\, y > 0\}$ with the local metric $ds^2 = \frac{dx^2+dy^2}{y}$.

The geodesics, i.e. the hyperbolic lines, are the half-circles in $I\!P$ centered on the *real axis* (the x-axis) (exceptionnally the straight half-lines parallel to the y-axis) (See Figure 16).

The (so called "hyperbolic") distance between two points $z, z' \in I\!P$ is, up to a constant factor, the logarithm of the (real) cross-ratio $(z, z', a, a') = \frac{z-a}{z-a'} \cdot \frac{z'-a'}{z'-a}$ where a, a' are the traces on the real axis of the geodesic through zz'.

The bisecting line σ of z and z' is the upper half of the circle Σ centered on the real axis and belonging to the pencil with limit points z and z'.

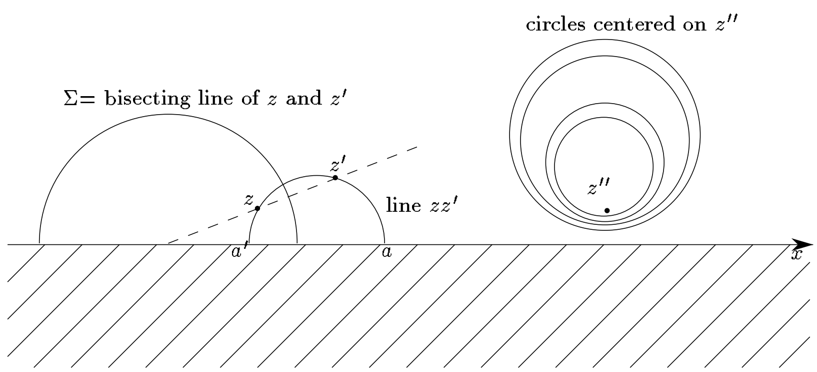

Figure 16: Hyperbolic lines and circles.

The hyperbolic circles centered on a given point z are the euclidean circles forming the pencil of circles having z as a limit point z and the real axis as radical axis.

Thus one can define in the usual way the hyperbolic Voronoi diagram of a finite set $\mathcal{S}$ of sites in $I\!P$. The cell of $z \in \mathcal{S}$ is the set of points nearer (in the hyperbolic sense) to z than to any other points in $\mathcal{S}$; this is a (hyperbolically) convex subset of $I\!P$, limited by arcs of bisecting lines, and the vertices are the centers (in the hyperbolic sense) of circles (in both hyperbolic and euclidean sense) passing through three sites of $\mathcal{S}$, and with empty interiors, that is to say the circles circumscribing an ordinary Delaunay triangle of $\mathcal{S}$ and entirely drawn inside $I\!P$. An example of such a diagram on five points is given in Figure 17.

As in ($\mathcal{P}$ *1*), the nearest neighbor of z is given by intersecting $U_{\mathcal{S}}$ with the line of σ representing the circles centered on z (in the hyperbolic sense). All such lines, for all $z \in I\!P$, have the same direction Δ, since they are associated to pencils with the same radical axis. Thus, if we project $U_{\mathcal{S}}$ in the direction of Δ onto the paraboloid Π, and

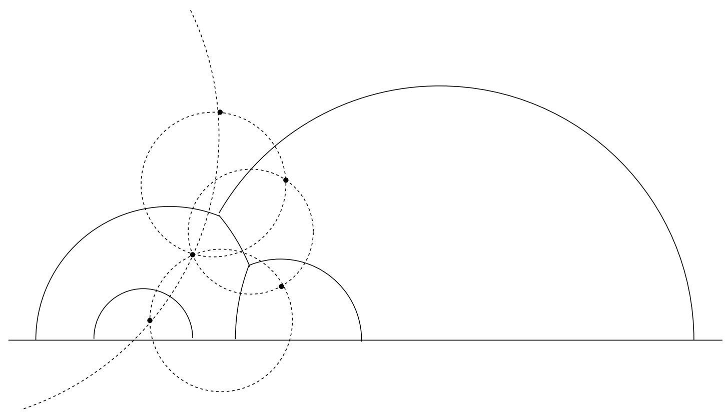

Figure 17: A Voronoi diagram in the hyperbolic Poincaré half plane

then orthogonally onto $I\!P$, we obtain the hyperbolic Voronoi diagram of $\mathcal{S}$: in fact, an edge of $U_{\mathcal{S}}$ projects onto Π in the Δ-direction on the set of point-spheres equidistant (in the hyperbolic sense) from two sites of $\mathcal{S}$. This gives a correspondence between the euclidean and the hyperbolic Voronoi diagrams.

Finally, notice that this can be generalized to higher dimensions, though we present it in the plane for the sake of simplicity.

5 Complexity and algorithms

This paper proves that any Voronoi diagram in d dimension can be transformed in an upper envelope of appropriate surfaces in dimension $d+1$. So all results concerning complexities or algorithms for upper envelopes can be applied to Voronoi diagrams. Unfortunately, there are only few known results. The algorithm of [SS90] can be applied to particular cases.

The Space of Spheres also allows us to derive algorithms which are not based on the direct use of upper envelopes. We present such algorithms in the following sections.

5.1 Order k diagrams

The algorithm given in [Aur90] for order k power diagrams generalizes to weighted diagrams. It allows to avoid the computations of the centroids of all k-tuples of $\mathcal{S}$, except for the relevant ones. If the order $k-1$ diagram has already been computed, a $(d-1)$-face of this diagram separates the region of $S_1, \ldots, S_{k-1}$ and the region of $S_1, \ldots, S_{k-2}, S_k$. Each such $(d-1)$-face produces a relevant k-tuple in the order k diagram, namely $\{S_1, \ldots, S_k\}$.

5.2 Output sensitive algorithms for constrained data

5.2.1 Voronoi diagrams for a constrained set of points

We only recall here the algorithm given in [BCDT91]. The point sites belong to a 3 dimensional space, and they are assumed to lie in k different planes $\mathcal{P}_i, i = 1, \ldots, k$, for some k in $I\!N$. Let $\mathcal{S}_i$ denote the set of sites lying on $\mathcal{P}_i$. The general idea is to start from one initial tetrahedron, and to construct the whole 3 dimensional Delaunay triangulation incrementally, by adding the neighbors of the already computed tetrahedra. The order in which the tetrahedra are added is given by the mechanism of shelling [Sei86,BM71].

The search for a neighbor $(ABCX)$ of a tetrahedron $(ABCD)$ through the face (ABC) reduces to a query problem in each of the k spaces of circles σ_i associated to the planes $\mathcal{P}_i$. The spheres circumscribing $(ABCD)$ and $(ABCX)$ are the two extremal empty spheres of the pencil of spheres with base points A, B, C. The intersection of this pencil of spheres with each plane $\mathcal{P}_i$ is a pencil of circles, that is a line in σ_i. The extremal empty circles of such a pencil are given by the intersections between this line and the convex $U_{\mathcal{S}_i}$ (see Section 4.1). It only remains to select the searched circle from the $2k$ found circles. The convex polytopes $U_{\mathcal{S}_i}, 1 \leq i \leq k$ are computed in a preprocessing step.

We can show the following theorem :

Theorem [BCDT91] *The three dimensional Delaunay triangulation of n points lying in k planes can be computed in $O(tk \log n)$ time and $O(n)$ extra space, where t is the size of the output.*

If $k = 2$, it is possible to identify σ_1 and σ_2 so that the algorithm only consists in superimposing in some sense the two convex $U_{\mathcal{S}_i}, i = 1, 2$, which yields :

Theorem [BCDT91] *The three dimensional Delaunay triangulation of n points lying in 2 planes can be computed in $O(t + n \log n)$ time and $O(n)$ space.*

These results have been generalized to higher dimensions, using some results by J. Matoušek [Mat91].

Theorem *Constructing the d dimensional Delaunay triangulation of n points constrained to belong to k p-subspaces can be done in time and space*

$$T = O\left(kn^{\gamma+\varepsilon}t'^{\gamma+\varepsilon} + kt''(\log n)^{O(1)}\right)$$

where ε is any positive constant,
t is the size of the output, $t' = \min(t, n^{\lfloor \frac{p+1}{2} \rfloor})$ and $t'' = t - t'$

$$\textit{and} \quad \gamma = \frac{1}{1 + \frac{1}{\lfloor \frac{p+1}{2} \rfloor}} \qquad \left(\gamma = \frac{2}{3} \textit{ for } p = 3, 4; \;\; \gamma = \frac{3}{4} \textit{ for } p = 5, 6 \ldots\right)$$

5.2.2 Power diagrams of a constrained set of spheres

The previous results can be generalized to the computation of the $(d+1)$ dimensional power diagram of a set $\mathcal{S}$ of d-spheres whose centers lie on k d-hyperplanes $\mathcal{P}_1, \ldots, \mathcal{P}_k$.

Let v be a vertex of the power diagram of $\mathcal{S}$. v is the center of a d-sphere V that is orthogonal to d-spheres $S_1, \ldots, S_{d+1}, S_{d+2}$ (the power of v with respect to each S_i is the square radius of V). The neighbor x of v incident to the edge of the power diagram consisting of points equidistant from $S_1, \ldots, S_{d+1}$ is the center of a d-sphere X orthogonal to d-spheres $S_1, \ldots, S_{d+1}, T$, for some searched d-sphere T. The supporting line of the edge (vx) corresponds to the pencil P of d-spheres orthogonal to $S_1, \ldots, S_{d+1}$.

Let $\mathcal{P}_i$ be the hyperplane containing the center of T. The intersection of P with $\mathcal{P}_i$ is a pencil of $(d-1)$-spheres P_i in $\mathcal{P}_i$, that is a line in σ_i. As T and X are orthogonal, the $(d-1)$-spheres $T \cap \mathcal{P}_i$ and $X \cap \mathcal{P}_i$ are orthogonal in $\mathcal{P}_i$. In other words, $X \cap \mathcal{P}_i$ is the $(d-1)$-sphere given by the intersection in σ_i of the pencil of $(d-1)$-spheres P_i (a line in σ_i) with $U_{\mathcal{S}_i}$, where $U_{\mathcal{S}_i}$ corresponds to the power diagram in $\mathcal{P}_i$ of the $(d-1)$-spheres $S \cap \mathcal{P}_i$, for all d-spheres $S \in \mathcal{S}_i$.

The three theorems of Section 5.2.1 can be generalized here.

Theorem *Constructing the d dimensional power diagram of n spheres whose centers are constrained to belong to k p-subspaces can be done in time and space*

$$T = O\left(kn^{\gamma+\varepsilon}t'^{\gamma+\varepsilon} + kt''(\log n)^{O(1)}\right)$$

where ε, γ, t' and t'' are defined as in Section 5.2.1.

If $d=3$ and $p=2$, the time complexity is $O(tk \log n)$ and the space required is $O(n)$. Furthermore, if $k=2$, the time complexity reduces to $O(t + n\log n)$.

5.2.3 Weighted power diagrams with a fixed number of weights

We have proved in Section 4.4 that there is a duality between the weighted power diagram of a set $\mathcal{S}$ of $(d-1)$-spheres in dimension d and a power diagram of a set $\{\sigma_P, P \in \mathcal{S}\}$ of d-spheres in dimension $d+1$, and that the last coordinate of σ_P only depends on the weight $w(P)$ (see Equation (6)).

The results given in the preceding section can be applied to the set of spheres $\{\sigma_P, P \in \mathcal{S}\}$, if the number of weights of the $(d-1)$-spheres of $\mathcal{S}$ is k.

This leads to an algorithm whose complexity is output sensitive in the size of the power diagram obtained in σ. Unfortunately, this algorithm is not output sensitive in the size of the weighted diagram in $\mathcal{P}$, as can be seen in the following example (Figure 18) :

$\mathcal{P}$ is the euclidean plane. δ is the distance as defined in Section 4.3. $\mathcal{S}$ consists in two kinds of circles. $\mathcal{S} = \{P_i, 1 \le i \le n\} \cup \{Q_j, 1 \le j \le n\}$. The centers of the circles $P_i = (\Phi_{P_i}, \langle \Phi_{P_i}, \Phi_{P_i}\rangle)$ lie on the unit circle α of $\mathcal{P}$ and their radii are zero. Thus $\langle \Phi_{P_i}, \Phi_{P_i}\rangle = 1$. Their weight is -4. From Section 4.4, the equation of a sphere σ_{P_i} is

$$(\sigma_{P_i}) \qquad 0 \;=\; \langle \Phi, \Phi\rangle + \chi^2 - \frac{2}{5}\langle \Phi, \Phi_{P_i}\rangle - \frac{4}{5}\chi + \frac{1}{5}$$

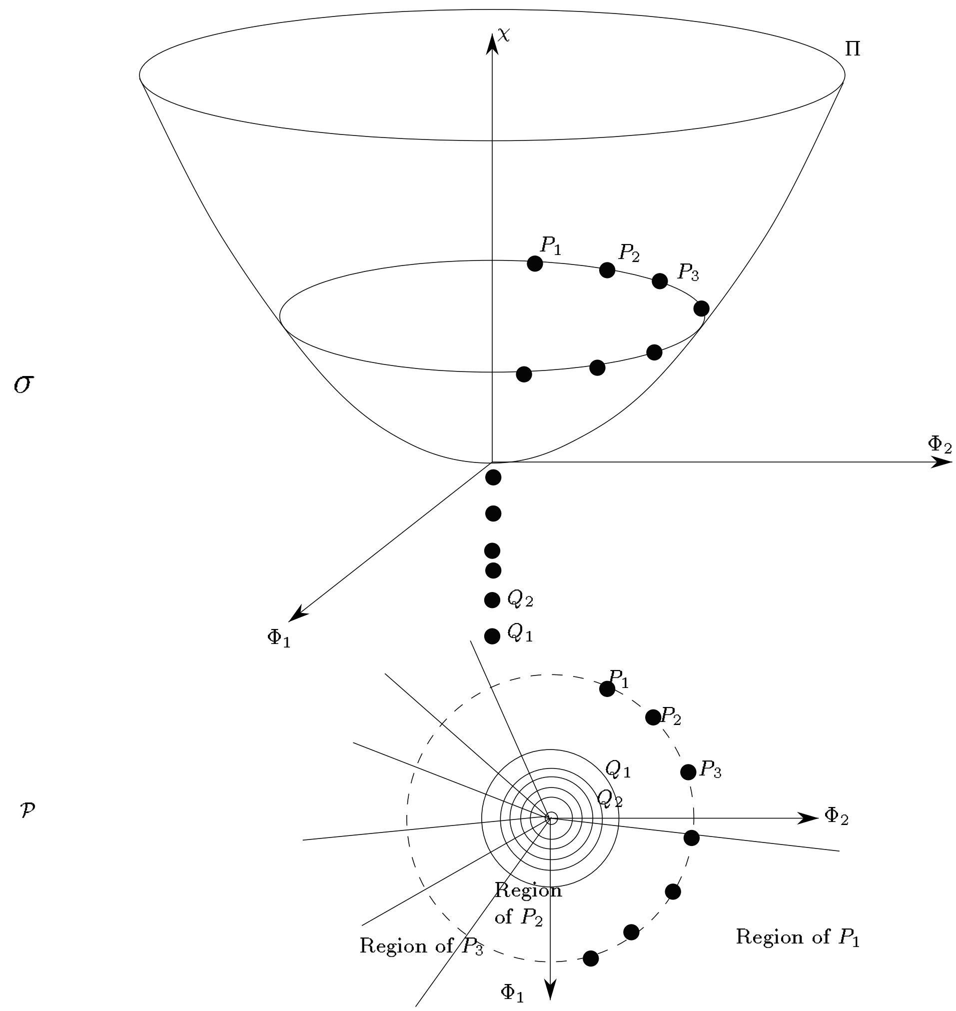

Figure 18: Linear weighted diagram mapped onto a quadratic power diagram

$$0 = \left\langle \Phi - \frac{1}{5}\Phi_{P_i}, \Phi - \frac{1}{5}\Phi_{P_i} \right\rangle + \left(\chi - \frac{2}{5}\right)^2$$

The spheres σ_{P_i} are points in the three dimensional space σ. They lie on an horizontal circle centered on the χ-axis.

The circles $Q_j = (0, \chi_{Q_j})$ are centered on the origin of $\mathcal{P}$. χ_{Q_j} and $w(Q_j)$ are computed so that σ_{Q_j} is a point in σ. From Section 4.4, the square radius of σ_{Q_j} is $\left(\frac{w(Q_j)}{2(w(Q_j)-1)}\right)^2 - \frac{\chi_{Q_j}}{1-w(Q_j)}$. So we take $w(Q_j) = \frac{2}{\chi_{Q_j}} + \sqrt{4 + \frac{4}{\chi_{Q_j}}}$. We choose $\chi_{Q_1} = -\frac{1}{3}$ and $\chi_{Q_j} \in \left[-\frac{1}{3}, 0\right]$

The spheres σ_{Q_j} are points in σ. They lie on the χ-axis.

The power diagram of the set $\{\sigma_{P_i}, 1 \leq i \leq n\} \cup \{\sigma_{Q_j}, 1 \leq j \leq n\}$ is known to be quadratic [PS85].

Let us now compute the weighted diagram of $\mathcal{S}$. We show that the regions of the sites $Q_j, 1 \leq j \leq n$ are empty in this diagram.

First note that Q_1 is the outermost of the circles $Q_j, 1 \leq j \leq n$. If M is a point of $\mathcal{P}$ exterior to circle Q_1, we have $\forall i, j, \delta(M, P_i) \leq 0 < \delta(M, Q_j)$, because the weights of the P_i's are negative and the weights of the Q_j's are positive. On the other hand, $\forall M, \forall j, \delta(M, Q_j) \geq \frac{\chi_{Q_j}}{w(Q_j)}$ (since the square radius of Q_j is $-\chi_{Q_j}$) which is minimal for $j = 1$, thus $\delta(M, Q_j) \geq -\frac{1}{30}$. If M lies in the interior of Q_1, its euclidean distance to the nearest P_i is greater than $1 - \sqrt{-\chi_{Q_1}}$, so $\forall i, \delta(M, P_i) \leq \frac{\left(1-\sqrt{-\chi_{Q_1}}\right)^2}{-4} < -\frac{1}{30}$. So, each point of $\mathcal{P}$, is nearer to P_i than Q_j, for all i, j.

Thus the weighted diagram of $\mathcal{S}$ is the usual furthest neighbor Voronoi diagram of the sites $P_i, 1 \leq i \leq n$, since the weights $w(P_i)$ are negative. Its size is linear.

We remark that our example is constructed with circles with positive and negative weights. It remains an open question whether an example for point sites can be found.

This example shows that the transformation given in [Aur87], that transforms a weighted power diagram in dimension d into a power diagram in dimension $d+1$, may increase the complexity of the problem.

6 Conclusion

We have introduced a geometric framework which allows an unification of many duality results for generalized Voronoi diagrams. In this framework, all results can be proven by only using purely geometric interpretations, without any analytic calculations. We can apply it to different kinds of Voronoi diagrams, and combine easily the results to hold weighted order k diagrams for the hyperbolic metric for example.

The idea of the space of circles or spheres can be used to solve other problems concerning circles and spheres : for example, the determination of the ring with minimal surface, defined by two cocircular circles, containing a given set of points, transforms in the space of spheres into a linear programming problem. Another space of circles is used in [AS91] to solve the intersections counting problem in a set of circles.

Acknowledgements

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